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Author(s): Morton Brown and Herman Gluck

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STABLE STRUCTURES ON MANIFOLDS: III APPLICATIONS

BY MORTON BROWN AND HERMAN GLUCK*

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1. Introduction

In this paper we apply the machinery of stable structures developed in [1] and [2] to some problems in the field of topological manifolds.

The first part considers the question of the existence of tubular neighborhoods of locally flat curves, the main result being that a necessary and sufficient condition for an arbitrary locally flat simple closed curve in a topological manifold to have a trivial tubular neighborhood is that the manifold support a stable structure. An interesting corollary is that a locally flat simple closed curve in an orientable combinatorial manifold is tame.

The second part provides a solution of the Schoenflies problem for $S^{n-1} \times S^1$. This solution implies that $S^{n-1} \times S^1$ is homogeneous, and that the stable structure on it is unique up to stable homeomorphism. It is also shown that the operation of adding a handle to a manifold is well-defined. The n -sphere with handles is therefore well-defined by the number of handles.

2. Definitions

The reader is first referred to the definitions supplied in [1] and [2].

The underlying space of an R^{n-1} bundle over S^1 will be called a *tube*. If T is a tube, then there is a homeomorphism $h: R^{n-1} \rightarrow R^{n-1}$ such that T is homeomorphic to the space obtained from $R^{n-1} \times [0, 1]$ by identifying $(x, 0)$ with $(h(x), 1)$ for all $x \in R^{n-1}$.

A tube will be called *trivial* if it is homeomorphic to $R^{n-1} \times S^1$. A subset C of the trivial tube T will be called a *core* of T if there is a homeomorphism $F: R^{n-1} \times S^1 \rightarrow T$ such that $C = F(0 \times S^1)$. In such a situation we will also say that C has the *trivial tubular neighborhood* T .

The homeomorphism f of a space X onto itself will be said to be *weakly isotopic* to the homeomorphism f' of X if there is a homeomorphism

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$F: X \times [0, 1] \rightarrow X \times [0, 1]$ such that, for all $x \in X$, $F(x, 0) = (f(x), 0)$ and $F(x, 1) = (f'(x), 1)$. Then F will be referred to as a *weak isotopy* between f and f' .

The manifold M^n will be said to be *homogeneous* if, for any two locally flat embeddings f and f' of the closed unit n -cell D^n into M^n , there is a homeomorphism h of M^n onto itself such that $hf_1 = f_2$.

If M_1 and M_2 are two n -manifolds, a *sum* $M_1 \# M_2$ is (ambiguously) obtained by removing the interiors of locally flat closed n -cells from M_1 and M_2 and attaching the new boundaries of the remaining manifolds to one another by some homeomorphism. If all sums of M_1 and M_2 are homeomorphic, we will say that $M_1 \# M_2$ is *well-defined*. The manifold $M_1 \# (S^{n-1} \times S^1)$ will be said to be obtained by adding a *handle* to M_1 .

A subset A of the combinatorial manifold M^n will be said to be *tame* in M^n if there is a homeomorphism of M^n onto itself which takes A onto a subcomplex of some rectilinear subdivision of M^n .

We close this section with a statement of the *Schoenflies problem* for n -dimensional manifolds.

SCHOENFLIES PROBLEM. *Let f_1 and f_2 be arbitrary locally flat embeddings of S^{n-1} into M^n . Find necessary and sufficient conditions for the existence of a homeomorphism h of M^n onto itself such that $hf_1 = f_2$.*

(I)

3. Statement of results

THEOREM 3.1. *Let M^n be a topological manifold of dimension > 2 and ρ a metric on M^n . If $f: S^1 \rightarrow M^n$ is a continuous map and $\varepsilon > 0$ a given positive number, there is a locally flat embedding $f': S^1 \rightarrow M^n$ such that, for all $x \in S^1$,*

$$\rho(f(x), f'(x)) < \varepsilon.$$

THEOREM 3.2. *Every tube in a stable manifold is trivial.*

THEOREM 3.3. *Let M^n be a connected topological manifold and $(T_i)_i$ a family of trivial tubes whose cores freely generate $\pi_1(M^n)$. Then M^n admits a stable structure.*

The main theorem of this part of the paper is

THEOREM 3.4. *Every locally flat simple closed curve in a stable manifold M^n has a trivial tubular neighborhood in M^n .*

Combining this with Theorems 3.1 and 3.3, we get the following characterizations of stable manifolds.

THEOREM 3.5. *M^n admits a stable structure if and only if $\pi_1(M^n)$ can be freely generated by trivial tubes.*

THEOREM 3.6. *M^n admits a stable structure if and only if every locally flat simple closed curve in M^n has a trivial tubular neighborhood in M^n .*

Theorem 3.5 gives the most practical way for recognizing stable manifolds.

Finally, Theorem 3.4, combined with a recent result of Homma, produces

THEOREM 3.7. *A locally flat simple closed curve in an orientable combinatorial manifold is tame.*

4. Proof of Theorem 3.1

Let M^n be a topological manifold of dimension > 2 , ρ a metric on M^n , $f: S^1 \rightarrow M^n$ a continuous map and $\varepsilon > 0$ a given positive number. We are required to find a locally flat embedding $f': S^1 \rightarrow M^n$ such that, for all $x \in S^1$, $\rho(f(x), f'(x)) < \varepsilon$.

If $\dim M^n = 3$, M^n can be triangulated as a combinatorial manifold, according to [3]. Do so, and let f' be a simplicial ε -approximation to f whose image is in general position. Since a polygonal curve in a combinatorial manifold is always locally flat, f' satisfies all the required conditions.

Assume, therefore, that $\dim M^n \geq 4$. Let $x_0, x_1, \dots, x_{k-1}, x_k, x_0$ be a partition of S^1 , and let $\widehat{x_i x_{i+1}}$ denote the 'smaller' closed arc of S^1 from x_i to x_{i+1} . The partition can be chosen so fine that, for each i , $f(\widehat{x_i x_{i+1}})$ is contained in an open n -cell $E_i \subset M^n$ of diameter less than ε . For each i , let $f'(x_i)$ be a point of $E_{i-1} \cap E_i$ chosen so that the various $f'(x_i)$ are all distinct. If the $f(x_i)$ are all distinct, we can put $f'(x_i) = f(x_i)$. The embedding f' will be constructed so that, for each i , $f'(\widehat{x_i x_{i+1}}) \subset E_i$. Since the diameter of E_i is less than ε , such an f' will automatically be an ε -approximation to f .

To begin, choose a combinatorial triangulation of E_0 and let $f'(\widehat{x_0 x_1})$ be a polygonal arc which runs in E_0 from $f'(x_0)$ to $f'(x_1)$, and which does not meet any other $f'(x_i)$.

Now look at E_1 . According to [4], there is a point $x_1^- \in \widehat{x_0 x_1}$ which lies so close to x_1 that E_1 has a combinatorial triangulation in which $f(\widehat{x_1^- x_1})$ appears as a polygonal arc. Then to get $f'(\widehat{x_1 x_2})$, just continue this polygonal arc over to $f'(x_2)$, taking care not to meet $f'(\widehat{x_0 x_1})$ or $f'(x_i)$ for $i \neq 1, 2$. Continuing in this way, we construct f' on $\widehat{x_0 x_1} \cup \widehat{x_1 x_2} \cup \dots \cup \widehat{x_{k-1} x_k}$.

Because $\dim M^n \geq 4$, there is, according to [5], a point $x_k^- \in \widehat{x_{k-1} x_k}$ so close to x_k and a point $x_0^+ \in \widehat{x_0 x_1}$ so close to x_0 that E_k has a combinatorial

triangulation in which both $f'(x_k^-x_k)$ and $f(x_0x_0^+)$ appear as polygonal arcs. Then f' is extended over x_kx_0 by connecting these two polygonal arcs with a polygonal arc $f'(x_kx_0)$ which meets the previously constructed arcs just at $f'(x_k)$ and $f'(x_0)$.

This completes the construction of the embedding f' , which is an ϵ -approximation to f . By the construction, each point of $f'(S^1)$ has a neighborhood U in M^n which has a combinatorial triangulation in which $U \cap f'(S^1)$ appears as a polygonal arc. As remarked earlier, polygonal curves in combinatorial manifolds are always locally flat, so that $f'(S^1)$ is locally flat in M^n . This completes the proof.

5. Proof of Theorem 3.2

Let h be a homeomorphism of R^{n-1} onto itself, and let the tube T be obtained from $R^{n-1} \times [0, 1]$ by identifying $(x, 0)$ with $(h(x), 1)$ for all $x \in R^{n-1}$. Let $F: R^{n-1} \times [0, 1] \rightarrow T$ denote the decomposition map.

LEMMA 5.1. *If h is weakly isotopic to the identity, then T is trivial.*

Let $H: R^{n-1} \times [0, 1] \rightarrow R^{n-1} \times [0, 1]$ be a weak isotopy of the identity with h . Then $H(x, 0) = (x, 0)$ and $H(x, 1) = (h(x), 1)$ for all $x \in R^{n-1}$. Then $FH: R^{n-1} \times [0, 1] \rightarrow T$ is one-one except that $FH(x, 0) = FH(x, 1)$. Hence T is trivial.

LEMMA 5.2. *If T supports a stable structure, then h is weakly isotopic to the identity.*

The universal covering space of T is homeomorphic to $R^{n-1} \times R^1 = R^n$, and a generating covering transformation τ has the property

$$\tau(x, t) = (h(x), t + 1)$$

for $x \in R^{n-1}$ and $t \in R^1$.

Since T supports a stable structure, τ is a stable homeomorphism by [2, Cor. 2 to Th. 17.1]. Let g denote the stable homeomorphism of $R^{n-1} \times R^1$ such that $g(x, t) = (x, t - 1)$ for $x \in R^{n-1}$ and $t \in R^1$. Then $g\tau$ is a stable homeomorphism of $R^{n-1} \times R^1$ such that

$$g\tau(x, t) = (h(x), t) .$$

For the moment, think of S^n as the one point compactification of $R^{n-1} \times R^1$ by the point ∞ , and $S^{n-1} \subset S^n$ as the one point compactification of $R^{n-1} \times 0$ by ∞ . Extend $g\tau$ to a homeomorphism $(\overline{g\tau})$ of S^n which takes ∞ onto itself.

By [2, Th. 6.5], the stability of $g\tau$ implies the stability of $(\overline{g\tau})$. Since $(\overline{g\tau})$ is invariant on S^{n-1} and does not interchange its complementary domains, $(\overline{g\tau})/S^{n-1}$ is weakly isotopic to the identity, according to

[1, Th. 8.3]. By [6, Th. 5.1], $g\tau/R^{n-1} \times 0$ must be weakly isotopic to the identity. Since $g\tau(x, 0) = (h(x), 0)$, so must h be weakly isotopic to the identity.

If the tube T lies in a stable manifold, then it inherits a stable structure, and must therefore be trivial by the two preceding lemmas. This proves Theorem 3.2.

6. Proof of Theorem 3.3

Let M^n be a connected topological manifold, and $(T_i)_I$ a family of trivial tubes whose cores *freely generate* $\pi_1(M^n)$. What we mean by this is that, if $(c_i)_I$ is a family of oriented cores of the tubes, then upon selecting a basepoint m in M^n and joining the curves c_i to m , the resulting curves represent a set of normal generators for $\pi_1(M^n, m)$. By pushing the tubes around in M^n , we may assume without loss of generality that each core c_i already passes through the basepoint m .

Let \tilde{M}^n denote the universal covering space of M^n with basepoint \tilde{m} lying over m , and $p: \tilde{M}^n \rightarrow M^n$ the projection map. Then to each oriented core c_i there corresponds a covering transformation τ_i of \tilde{M}^n . We claim that τ_i must be a stable homeomorphism.

Let \tilde{T}_i denote the component of $p^{-1}(T_i)$ containing \tilde{m} . Then $\tau_i(\tilde{T}_i) = \tilde{T}_i$. T_i , being trivial, is certainly stable. Hence τ_i/\tilde{T}_i must be stable by [2, Cor. 2 to Theorem 17.1]. Then τ_i must be stable by [2, Th. 6.5].

Since the family $(c_i)_I$ represents a set of normal generators for $\pi_1(M^n, m)$, $(\tau_i)_I$ is a set of normal generators for the group of all covering transformations of \tilde{M}^n . Therefore all covering transformations of \tilde{M}^n must be stable, and M^n then admits a stable structure by [2, Th. 17.3].

7. A lemma

Let C denote the subset

$$(D^{n-1} \times [0, 1]) \cup (0 \times [1, 6]) \cup (D^{n-1} \times [6, 7])$$

of $D^{n-1} \times [0, 7]$. An embedding f of C into R^n will be said to be *locally flat* if

- (i) it is locally flat in the ordinary sense at all points of C other than $(0, 1)$ and $(0, 6)$, and
- (ii) both $f(0, 1)$ and $f(0, 6)$ have neighborhoods U in R^n such that the pair $(U, U \cap f(C))$ is homeomorphic to the pair

$$(R^n, (R^{n-1} \times R_+^1) \cup (0 \times R^1)),$$

where R_+^1 denotes the set of non-negative real numbers.

Note that $C \subset D^{n-1} \times [0, 7] \subset R^{n-1} \times R^1 = R^n$, hence we may speak of the *inclusion* $C \subset R^n$.

LEMMA 7.1. *Let f be a locally flat embedding of C into R^n . Then there is a homeomorphism g of R^n onto itself such that gf is the inclusion $C \subset R^n$ if and only if there is a stable homeomorphism h of R^n onto itself such that*

$$hf(x, t) = f(x, t + 6)$$

for all $x \in D^{n-1}$ and $t \in [0, 1]$.

First suppose that such a g exists, and define $\tau : R^{n-1} \times R^1 \rightarrow R^{n-1} \times R^1$ by $\tau(x, t) = (x, t + 6)$. Then τ is a stable homeomorphism of $R^{n-1} \times R^1 = R^n$. Let $h = g^{-1}\tau g$, which is stable because τ is. Then

$$hf(x, t) = g^{-1}\tau gf(x, t) = g^{-1}\tau(x, t) = g^{-1}(x, t + 6) = f(x, t + 6),$$

for all $x \in D^{n-1}$ and $t \in [0, 1]$.

Suppose, on the other hand, that such an h exists. If $n = 2$, the existence of g follows from the classical Schoenflies theorem [7], so we assume $n > 2$.

It follows from [1, Th. 10.4] that there is a homeomorphism g_1 of R^n onto itself such that $g_1f/(D^{n-1} \times [0, 1]) \cup (D^{n-1} \times [6, 7])$ is the inclusion. It remains to 'straighten out' $g_1f(0 \times [1, 6])$.

Using the local flatness of f at $(0, 1)$ and $(0, 6)$, we can find a homeomorphism g_2 of R^n onto itself such that the restriction of g_2g_1f to the set

$$(D^{n-1} \times [0, 1]) \cup (0 \times [1, 3]) \cup (0 \times [4, 6]) \cup (D^{n-1} \times [6, 7])$$

is the inclusion. Now only $g_2g_1f(0 \times [3, 4])$ has to be straightened out.

Using the local flatness of g_2g_1f , we can find an open n -cell $U \subset R^n$ such that

- (i) $U \cap g_2g_1f(C) = g_2g_1f(0 \times [2, 5]) = \alpha$,
- (ii) the pair (U, α) is homeomorphic to the pair (R^n, R^1) .

It is then possible to find a homeomorphism g_3 of U onto itself such that

- (i) g_3 restricts to the identity near the boundary of U ,
- (ii) $g_3(\alpha)$ is polygonal in R^n .

The existence of g_3 follows from [3] and [10] when $n = 3$, and from [8, Th. 2.1], a modification of Homma's theorem [5], when $n \geq 4$.

Extend g_3 *via* the identity to a homeomorphism g_3 of R^n onto itself. Then

- (i) $g_3g_2g_1f/(D^{n-1} \times [0, 1]) \cup (D^{n-1} \times [6, 7])$ is the inclusion,
- (ii) $g_3g_2g_1f(0 \times [0, 7])$ is polygonal.

Now it is easy to get a homeomorphism g_4 of R^n onto itself such that

$$g_4g_3g_2g_1f|C$$

is the inclusion. This is a standard technique for $n = 3$, and follows

from a general position argument for $n \geq 4$.

Putting $g = g_3 g_2 g_1$ completes the proof.

COROLLARY. *If f is a locally flat embedding of C into R^n and h a stable homeomorphism of R^n onto itself such that $hf(x, t) = f(x, t + 6)$ for $x \in D^{n-1}$ and $t \in [0, 1]$, then there exists a locally flat embedding $F: D^{n-1} \times [0, 7] \rightarrow R^n$ which extends f .*

For if $i: D^{n-1} \times [0, 7] \subset R^n$ is the inclusion and g is obtained from the above lemma, simply put $F = g^{-1}i$.

8. Proof of Theorem 3.4

Let S be a locally flat simple closed curve in the stable manifold M^n . It is shown in [8] that S may be written as the union of two open arcs A and A' which have neighborhoods U_A and $U_{A'}$ in M^n such that

- (i) $U_A \cap S = A$ and (U_A, A) is homeomorphic to (R^n, R^1) ,
- (ii) $U_{A'} \cap S = A'$ and $(U_{A'}, A')$ is homeomorphic to (R^n, R^1) .

Choose points p, p' from one component of $A \cap A'$ and q, q' from the other component in such a way that if a denotes the closed subarc of A with endpoints p and q , and a' the closed subarc of A' with endpoints p' and q' , then the arcs a and a' overlap at both ends.

It follows from (i) above that there is a locally flat embedding F of $D^{n-1} \times [0, 7]$ into U_A such that

- (i) $F(D^{n-1} \times [0, 7]) \cap S = F(0 \times [0, 7]) = \alpha$,
- (ii) $F(0, 0) = p, F(0, 1) = p', F(0, 6) = q'$ and $F(0, 7) = q$,
- (iii) $F(D^{n-1} \times [0, 1])$ and $F(D^{n-1} \times [6, 7])$ both lie in $U_A \cap U_{A'}$,
- (iv) $F(D^{n-1} \times [0, 7]) \cup A$ is also locally flat.

It is then easy to construct a stable homeomorphism g of M^n onto itself which restricts to the identity outside a neighborhood of $F(D^{n-1} \times [0, 7])$, such that for $x \in D^{n-1}$ and $t \in [0, 1]$,

$$gF(x, t) = F(x, t + 6) .$$

Then by [2, Th. 14.1], since M^n is stable, there is a stable homeomorphism h of $U_{A'}$ onto itself, with compact support, such that for $x \in D^{n-1}$ and $t \in [0, 1]$,

$$hF(x, t) = F(x, t + 6) .$$

That is, h and g coincide on $F(D^{n-1} \times [0, 1])$.

Now define a locally flat embedding $f': C \rightarrow U_{A'}$ as follows:

- (i) $f'(x, t) = F(x, 1 - t)$ for $x \in D^{n-1}$ and $t \in [0, 1]$,
- (ii) $f'(x, t) = F(x, 13 - t)$ for $x \in D^{n-1}$ and $t \in [6, 7]$,
- (iii) $f'(0 \times [0, 7]) = \alpha'$.

By the local flatness of F and condition (iv) above, f' actually is locally flat. Furthermore, for $x \in D^{n-1}$ and $t \in [0, 1]$,

$$\begin{aligned} hf'(x, t) &= hF(x, 1 - t) = F(x, 7 - t) = F(x, 13 - (t + 6)) \\ &= f'(x, t + 6). \end{aligned}$$

Then by the Corollary to Lemma 7.1, f' can be extended to a locally flat embedding $F' : D^{n-1} \times [0, 7] \rightarrow U_A$. By compressing $F'(D^{n-1} \times [1, 6])$, we can insure that it is disjoint from $F(D^{n-1} \times [1, 6])$.

Finally, we define a map

$$H : R^{n-1} \times [0, 12] \rightarrow M^n$$

as follows. Let $r : R^{n-1} \rightarrow \text{Int } D^{n-1}$ be a radial contraction. Then let

$$\begin{aligned} H(x, t) &= F(r(x), t) && \text{for } t \in [0, 6] \\ H(x, t) &= F'(r(x), 13 - t) && \text{for } t \in [6, 12]. \end{aligned}$$

Then

$$H(x, 6) = F(r(x), 6) = F'(r(x), 7) = H(x, 6),$$

so H is well-defined. Furthermore

$$H(x, 12) = F'(r(x), 1) = F(r(x), 0) = H(x, 0),$$

but H is one-one otherwise. Thus $T = H(R^{n-1} \times [0, 12])$ is a trivial tube in M^n with core $S = H(0 \times [0, 12])$, completing the proof of Theorem 3.4.

9. Proof of Theorems 3.5 and 3.6

To prove Theorem 3.5, half of which is already contained in Theorem 3.3, we start with a stable manifold M^n . If $n = 2$, it is well known that $\pi_1(M^n)$ can be freely generated by trivial tubes. If $n > 2$, Theorem 3.1 says that $\pi_1(M^n)$ can be generated by locally flat simple closed curves, while Theorem 3.4 encloses such curves in trivial tubes.

To prove Theorem 3.6, half of which is contained in Theorem 3.4, we start with a manifold M^n in which every locally flat simple closed curve has a trivial tubular neighborhood. We can assume $n > 2$, for all orientable 2-manifolds are stable. Then by Theorem 3.1, $\pi_1(M^n)$ can be generated by locally flat simple closed curves, and therefore by trivial tubes. Hence M^n is stable by Theorem 3.3.

10. Proof of Theorem 3.7

Let S be a locally flat simple closed curve in the orientable combinatorial manifold M^n . By [2, Th. 10.4], M^n admits a stable structure. Hence by Theorem 3.4, there is a trivial tube T in M^n containing S as a

core. There is obviously a combinatorial triangulation of T (independent of that of M^n) in which S appears as a subcomplex. Homma's theorem [5] then asserts that if $\dim n \geq 4$, there is, for any $\varepsilon > 0$, an ε -homeomorphism of M^n onto itself which takes S onto a polygonal curve.

If $n \leq 3$, the theorem is well known, [7] and [10].

(II)

11. Statement of results

THEOREM 11.1. *Let M_1 and M_2 be connected topological n -manifolds. If either M_1 or M_2 is homogeneous, then $M_1 \# M_2$ is well-defined. If both are homogeneous, then $M_1 \# M_2$ is also homogeneous.*

The next theorem provides a solution of the Schoenflies problem for $S^{n-1} \times S^1$.

THEOREM 11.2. *Let f_1 and f_2 be locally flat embeddings of S^{n-1} into $S^{n-1} \times S^1$ whose images either both separate or both do not separate $S^{n-1} \times S^1$. Then there is a homeomorphism h of $S^{n-1} \times S^1$ onto itself such that $hf_1 = f_2$.*

The following are then easy corollaries.

THEOREM 11.3. *$S^{n-1} \times S^1$ is homogeneous, and the stable structure on it is unique up to stable homeomorphism.*

THEOREM 11.4. *The operation of adding a handle to a connected topological manifold is well-defined.*

THEOREM 11.5. *The n -sphere with k -handles is well-defined and homogeneous, and the stable structure on it is unique up to stable homeomorphism.*

12. Sums of manifolds

In place of the definition of a *sum* of two manifolds given in § 2, we will find the following description more convenient.

Let f_1 be an embedding of R^n into the connected n -manifold M_1 and f_2 an embedding of R^n into the connected n -manifold M_2 . If $x \in R^n - 0$, let \bar{x} denote the conjugate of x through the unit $n - 1$ sphere S^{n-1} . Then from the disjoint union

$$(M_1 - f_1(0)) \cup (M_2 - f_2(0)),$$

we form a decomposition space $S(M_1, M_2; f_1, f_2)$ by identifying

$$f_1(x) \text{ with } f_2(\bar{x})$$

for each $x \in R^n - 0$.

Such a decomposition space is certainly homeomorphic to a sum of M_1 and M_2 in the sense of § 2. Since any locally flat embedding of D^n into an n -manifold can be extended to an embedding of R^n by [4] and [9], it follows that any sum of M_1 and M_2 in the sense of § 2 can be obtained as above.

Then $M_1 \# M_2$ is well-defined if and only if the topological type of $S(M_1, M_2; f_1, f_2)$ does not depend on f_1 and f_2 .

REMARK 1. If h_1 is a homeomorphism of M_1 onto itself and h_2 a homeomorphism of M_2 onto itself, then $S(M_1, M_2; f_1, f_2)$ is homeomorphic to $S(M_1, M_2; h_1f_1, h_2f_2)$.

REMARK 2. If the embeddings f_1 and g_1 of R^n into M_1 agree on the ball of radius r and the embeddings f_2 and g_2 of R^n into M_2 agree on the ball of radius $1/r$, then $S(M_1, M_2; f_1, f_2)$ is homeomorphic to $S(M_1, M_2; g_1, g_2)$.

LEMMA 12.1. *Let f_1 and g_1 be embeddings of R^n into the connected n -manifold M_1 and f_2 an embedding of R^n into the connected n -manifold M_2 . Then there is an embedding g_2 of R^n into M_2 such that $S(M_1, M_2; f_1, f_2)$ is homeomorphic to $S(M_1, M_2; g_1, g_2)$. Furthermore, if $f_1(R^n) = g_1(R^n)$ and $f_1(0) = g_1(0)$, then g_2 can be chosen so that $S(M_1, M_2; f_1, f_2) = S(M_1, M_2; g_1, g_2)$.*

First assume that $f_1(R^n) = g_1(R^n)$ and $f_1(0) = g_1(0)$. Define $g_2 : R^n \rightarrow M_2$ by

$$\begin{aligned} g_2(0) &= f_2(0) \\ g_2(x) &= f_2(\overline{f_1^{-1}g_1(x)}) \end{aligned} \quad \text{for } x \in R^n - 0.$$

To form $S(M_1, M_2; g_1, g_2)$, we identify $g_1(x)$ with $g_2(\overline{x}) = f_2(\overline{f_1^{-1}g_1(x)})$, which in forming $S(M_1, M_2; f_1, f_2)$ was identified with $f_1(\overline{f_1^{-1}g_1(x)}) = g_1(x)$. Therefore the two decomposition spaces, $S(M_1, M_2; f_1, f_2)$ and $S(M_1, M_2; g_1, g_2)$, must be equal.

In the general case, there is a homeomorphism h_1 of M_1 onto itself such that $h_1g_1(D^n) \subset f_1(R^n)$ and $h_1g_1(0) = f_1(0)$. By [4] and [9], h_1g_1/D^n extends to a homeomorphism $g'_1 : \rightarrow f_1(R^n) \subset M_1$.

Since $g'_1(R^n) = f_1(R^n)$ and $g'_1(0) = f_1(0)$, it follows from above that there is an embedding g_2 of R^n into M_2 such that $S(M_1, M_2; f_1, f_2) = S(M_1, M_2; g'_1, g_2)$. By Remark 2 above, $S(M_1, M_2; g'_1, g_2)$ is homeomorphic to $S(M_1, M_2; h_1g_1, g_2)$, which is homeomorphic to $S(M_1, M_2; g_1, g_2)$ by Remark 1. Hence $S(M_1, M_2; f_1, f_2)$ is homeomorphic to $S(M_1, M_2; g_1, g_2)$.

13. Proof of Theorem 11.1

We suppose that M_2 is homogeneous and must show that $S(M_1, M_2;$

f_1, f_2) is homomorphic to $S(M_1, M_2; g_1, g_2)$. By Lemma 12.1, there is an embedding g'_2 of R^n into M_2 such that $S(M_1, M_2; f_1, f_2)$ is homeomorphic to $S(M_1, M_2; g_1, g_2)$. Since M_2 is homogeneous, there is a homeomorphism h_2 of M_2 onto itself such that

$$h_2 g_2 / D^n = g_2 / D^n .$$

Then by Remark 1 of § 12, $S(M_1, M_2; g_1, g_2)$ is homeomorphic to $S(M_1, M_2; g_1, h_2 g_2)$, which is homeomorphic to $S(M_1, M_2; g_1, g_2)$ by Remark 2 of that section. Thus $S(M_1, M_2; f_1, f_2)$ is homeomorphic to $S(M_1, M_2; g_1, g_2)$, so $M_1 \# M_2$ is well-defined.

We now suppose that both M_1 and M_2 are homogeneous and must show that $M_1 \# M_2 = S(M_1, M_2; f_1, f_2)$ is homogeneous.

Let f and f' be arbitrary elements of $\text{Hom}(D^n, M_1 \# M_2)$. By Lemma 3.1 of [2], we can assume that both $f(D^n)$ and $f'(D^n)$ lie in $M_1 - f_1(D^n)$. Then f and f' may also be considered as elements of $\text{Hom}(D^n, M_1)$. Since M_1 is homogeneous, there is a homeomorphism h_1 of M_1 onto itself such that $h_1 f = f'$. We can easily arrange that $h_1 f_1(D^n)$ lies in the interior of $f_1(D^n)$ and that $h_1 f_1(0) = f_1(0)$.

According to [4] and [9], there is a homeomorphism $g_1: R^n \rightarrow f_1(R^n) \subset M_1$ such that

$$\begin{aligned} g_1(0) &= f_1(0) \\ g_1 / D^n &= h_1 f_1 / D^n . \end{aligned}$$

Define $g_2: R^n \rightarrow M_2$ by

$$\begin{aligned} g_2(0) &= f_2(0) \\ g_2(x) &= f_2(\overline{f_1^{-1} g_1(x)}) \end{aligned} \quad \text{for } x \in R^n - 0 .$$

By Lemma 12.1, $S(M_1, M_2; f_1, f_2) = S(M_1, M_2; g_1, g_2)$.

Since M_2 is homogeneous, there is a homeomorphism h_2 of M_2 onto itself such that $h_2 f_2 / D^n = g_2 / D^n$.

Finally define the homeomorphism h of $M_1 \# M_2$ onto itself by

$$\begin{aligned} h(m) &= h_1(m) && \text{if } m \in M_1 - \text{Int } f_1(D^n) \\ h(m) &= h_2(m) && \text{if } m \in M_2 - \text{Int } f_2(D^n) . \end{aligned}$$

If $x \in S^{n-1}$, then $h f_1(x) = h_1 f_1(x) = g_1(x)$, which corresponds in the decomposition to $g_2(x) = h_2 f_2(x) = h f_2(x)$. Since $f_1(x)$ and $f_2(x)$ also correspond in the decomposition, h is well-defined. Then $h f = h_1 f = f'$, so $M_1 \# M_2$ is homogeneous.

14. Proof of Theorem 11.2

Since Theorem 11.2 is well known for $n = 2$, we will assume that

$n \geq 3$, so that S^{n-1} will be simply connected. It will also be convenient to identify S^1 with the set of reals mod 1.

Since $n \geq 3$, the universal covering space of $S^{n-1} \times S^1$ is $S^{n-1} \times R^1$. Let $p: S^{n-1} \times R^1 \rightarrow S^{n-1} \times S^1$ be the projection map and $\tau: S^{n-1} \times R^1 \rightarrow S^{n-1} \times R^1$ the covering transformation defined by

$$\tau(x, t) = (x, t + 1) .$$

Note that the two-point (Freudenthal) compactification of $S^{n-1} \times R^1$ is homeomorphic to S^n . Then with any homeomorphism of $S^{n-1} \times R^1$ we can associate a homeomorphism of S^n which either leaves the north and south poles fixed, or interchanges them. By [2, Th. 65], stable homeomorphisms of $S^{n-1} \times R^1$ will be associated with stable homeomorphisms of S^n . This association will afford us the use of the machinery of [1].

LEMMA 14.1. *Let f be a locally flat embedding of S^{n-1} into $S^{n-1} \times S^1$ whose image does not separate $S^{n-1} \times S^1$. Then there is a homeomorphism h of $S^{n-1} \times S^1$ onto itself such that, for all $x \in S^{n-1}$,*

$$h(x, 0) = f(x) .$$

Since S^{n-1} is simply connected, there is a locally flat embedding \tilde{f} of S^{n-1} into $S^{n-1} \times R^1$ such that $p\tilde{f} = f$. Now \tilde{f} and $\tau\tilde{f}$ have disjoint images and similar orientations (as defined in [1, § 3]). Since τ is stable, [1, Ths. 5.4, 3.5] assert the existence of an embedding $G: S^{n-1} \times [0, 1] \rightarrow S^{n-1} \times R^1$ such that, for all $x \in S^{n-1}$,

$$\begin{aligned} G(x, 0) &= \tilde{f}(x) \\ G(x, 1) &= \tau\tilde{f}(x) . \end{aligned}$$

The desired homeomorphism h is then defined by

$$h(x, t) = pG(x, t)$$

for $x \in S^{n-1}$ and $t \in [0, 1]$ mod 1. Since

$$h(x, 1) = pG(x, 1) = p\tau\tilde{f}(x) = p\tilde{f}(x) = pG(x, 0) = h(x, 0) ,$$

h is well-defined, and since τ is a generating covering transformation, h is homeomorphism of $S^{n-1} \times S^1$ onto itself. Finally,

$$h(x, 0) = pG(x, 0) = p\tilde{f}(x) = f(x) ,$$

and the lemma is proved.

Now if f_1 and f_2 are locally flat embeddings of S^{n-1} into $S^{n-1} \times S^1$, neither of whose images separates $S^{n-1} \times S^1$, let h_1 and h_2 be homeomorphisms of $S^{n-1} \times S^1$ onto itself which satisfy the conditions of the above lemma. Then $h = h_2 h_1^{-1}$ is a homeomorphism of $S^{n-1} \times S^1$ onto itself

such that

$$hf_1 = f_2 .$$

This proves Theorem 11.2 for the non-separating case.

LEMMA 14.2. *Let f be a locally flat embedding of S^{n-1} into $S^{n-1} \times S^1$ whose image separates $S^{n-1} \times S^1$. Then f can be extended to an embedding of D^n into $S^{n-1} \times S^1$.*

Let $\tilde{f}: S^{n-1} \rightarrow S^{n-1} \times R^1$ cover f . Since $f(S^{n-1})$ separates $S^{n-1} \times S^1$, $\tilde{f}(S^{n-1})$ can not separate the ends of $S^{n-1} \times R^1$. Since $S^{n-1} \times R^1$ is homeomorphic to $R^n - 0$, it then follows from [4] and [9] that \tilde{f} can be extended to an embedding $\tilde{f}: D^n \rightarrow S^{n-1} \times R^1$.

Note that a covering transformation of $S^{n-1} \times R^1$ other than the identity cannot have a fixed point. Then since the various images of $\tilde{f}(S^{n-1})$ under the positive and negative powers of τ are disjoint, it follows from the Brouwer Fixed Point theorem that this must also be true of the various images of $\tilde{f}(D^n)$. Hence $p\tilde{f}: D^n \rightarrow S^{n-1} \times S^1$ is an embedding which extends f and proves the lemma.

Now let f_1 and f_2 be locally flat embeddings of S^{n-1} into $S^{n-1} \times S^1$, each of whose images separates $S^{n-1} \times S^1$. According to Lemma 14.2, f_1 and f_2 admit extensions, still denoted by f_1 and f_2 , to embeddings of D^n into $S^{n-1} \times S^1$. We will construct a homeomorphism h of $S^{n-1} \times S^1$ onto itself such that

$$hf_1 = f_2 .$$

This will not only complete the proof of Theorem 11.2, but will also demonstrate the homogeneity of $S^{n-1} \times S^1$.

It will be convenient for descriptive purposes to identify $S^{n-1} \times R^1$ with $R^n - 0$ in a fixed, but arbitrary, way.

Let \tilde{f}_1 and \tilde{f}_2 be liftings of f_1 and f_2 to embeddings of D^n into $S^{n-1} \times R^1 \subset R^n$. According to [1, Th. 6.1], there are locally flat embeddings \tilde{g}_1 and \tilde{g}_2 of S^{n-1} into $S^{n-1} \times (0, 1) \subset S^{n-1} \times R^1 \subset R^n$ such that

- (i) \tilde{g}_i is annularly equivalent to \tilde{f}_i/S^{n-1} in R^n ,
- (ii) $\tilde{g}_i(S^{n-1})$ separates the ends of $S^{n-1} \times R^1$.

Then $g_1 = p\tilde{g}_1$ and $g_2 = p\tilde{g}_2$ are locally flat embeddings of S^{n-1} into $S^{n-1} \times S^1$ whose images do not separate $S^{n-1} \times S^1$.

Let h_1 be a homeomorphism of $S^{n-1} \times S^1$ onto itself such that $h_1g_1 = g_2$, and \tilde{h}_1 a covering homeomorphism of $S^{n-1} \times R^1$ onto itself such that $\tilde{h}_1\tilde{g}_1 = \tilde{g}_2$. Then $\tilde{h}_1\tilde{f}_1$ and \tilde{f}_2 are embeddings of D^n into $S^{n-1} \times R^1$ which are annularly equivalent in R^n .

Let h_2 be a stable homeomorphism of $S^{n-1} \times S^1$ onto itself such that

$h_2 h_1 f_1(D^n)$ lies in the interior of $f_2(D^n)$, and \tilde{h}_2 a covering homeomorphism of $S^{n-1} \times R^1$ onto itself such that $\tilde{h}_2 \tilde{h}_1 \tilde{f}_1(D^n)$ lies in the interior of $\tilde{f}_2(D^n)$. Then \tilde{h}_2 is also stable. Hence $\tilde{h}_2 \tilde{h}_1 \tilde{f}_1$ is still annularly equivalent to \tilde{f}_2 in R^n . By [1, Th. 3.5 (i)], $\tilde{h}_2 \tilde{h}_1 \tilde{f}_1 \sim_A \tilde{f}_2$. Then $h_2 h_1 f_1 \sim_A f_2$, hence by [2, Th. 5.2] there is a stable homeomorphism h_3 of $S^{n-1} \times S^1$ onto itself such that

$$h_3 h_2 h_1 f_1 = f_2 .$$

Putting

$$h = h_3 h_2 h_1$$

then completes the proof of Theorem 11.2 and shows, as well, that $S^{n-1} \times S^1$ is homogeneous

According to [2, Th. 19.1], the stable structure on a homogeneous stable manifold is unique up to stable homeomorphism. We thus obtain Theorem 11.3.

Since $S^{n-1} \times S^1$ is homogeneous, Theorems 11.4 and 11.5 follow immediately from Theorem 11.1.

UNIVERSITY OF MICHIGAN AND INSTITUTE FOR ADVANCED STUDY,
INSTITUTE FOR ADVANCED STUDY AND HARVARD UNIVERSITY

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